*United Kingdom*
**Debt**
**Management**
**Office**

Cheapside House
138 Cheapside
London EC2V 6BB

# UNITED KINGDOM

# DEBT MANAGEMENT OFFICE

# The DMO's Yield Curve Model

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## Introduction

The most commonly used measure of a bond's return is the *gross redemption yield* – the single rate that, if used to discount each of the bond's cash flows individually, equates the bond's total present value to its price in the market. Implicit in this definition is the assumption that it will be possible to re-invest all of the bond's future coupon payments at the current redemption yield - clearly an unrealistic assumption. Using redemption yields to discount bond cash flows has the disadvantage that there is not a unique discount rate for a given maturity. Given this, it is more desirable to look at zero-coupon yields.

The *zero-coupon rate* for a given maturity is the rate at which an individual cash flow on this future date is discounted to determine its value today and can be thought of as the yield to maturity of a zero-coupon bond. The *zero-coupon yield curve* is simply the continuous curve of zero-coupon rates. When calculating the net present value of a bond's cash flows using the zero-coupon curve, a different zero-coupon rate is used for each cash flow. Across the market, all cash flows on a given date - irrespective of which bond they originate from - are discounted using the same zero-coupon rate. This article examines the method used by the DMO to estimate the zero-coupon gilt yield curve.

## Types of yield curve

Once estimated, the zero-coupon yield curve can be transformed uniquely into three other curves: the par yield curve, the discount function and the implied forward rate curve. Since the zero-coupon yield curve is not representative of the observed yield on a coupon-paying bond it is sometimes useful to look at the par yield curve instead. A coupon-paying bond is said to be priced at *par* if its current market price is equal to its face value.

The *par yield* at a given maturity represents the coupon required on a (hypothetical) coupon-paying bond of that maturity to ensure that it is trading at par. The *discount function* at a maturity *t* represents the value today of £1 repayable in *t* years' time. The implied forward rate curve consists of future one-period interest rates implied from the zero-coupon curve. It contains the same information as the zero-coupon curve but, because it is in effect a marginal curve (whereas the zero-coupon curve gives an average of expected rates over the chosen horizon), it shows the curve in a more detailed fashion.

Since the zero-coupon curve, par curve, discount function and forward curve are all unique transformations of each other, if it is possible to obtain or estimate rates for one of the curves, these rates can be transformed to give the other curves.

Figure 1 shows the discount function from the DMO's yield curve model for 31 March 2000 and Figure 2 illustrates the corresponding zero curve, par curve and implied forward rate curve.

**Figure 1: Discount function for 31 March 2000**

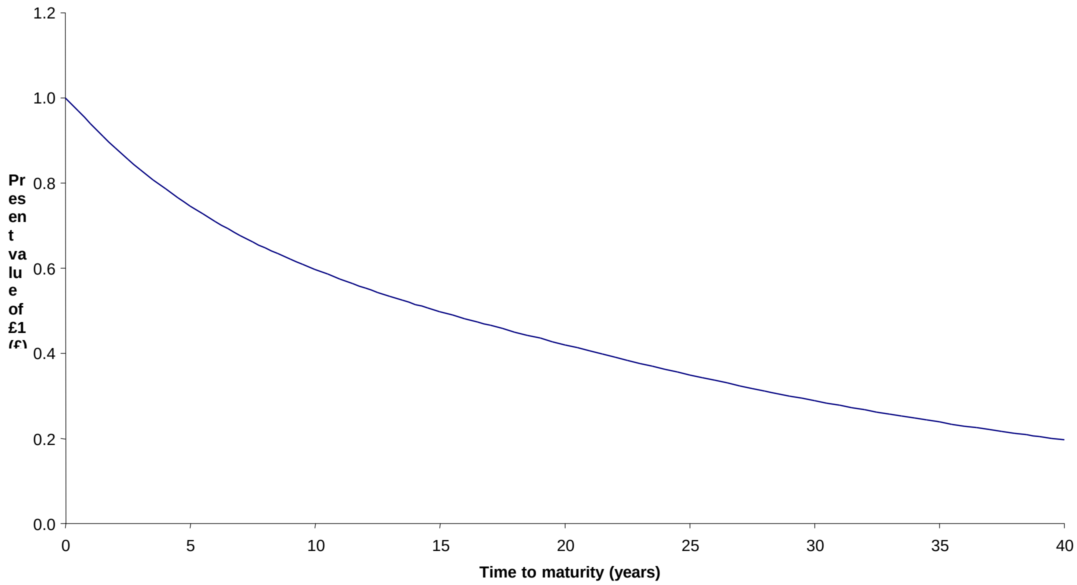
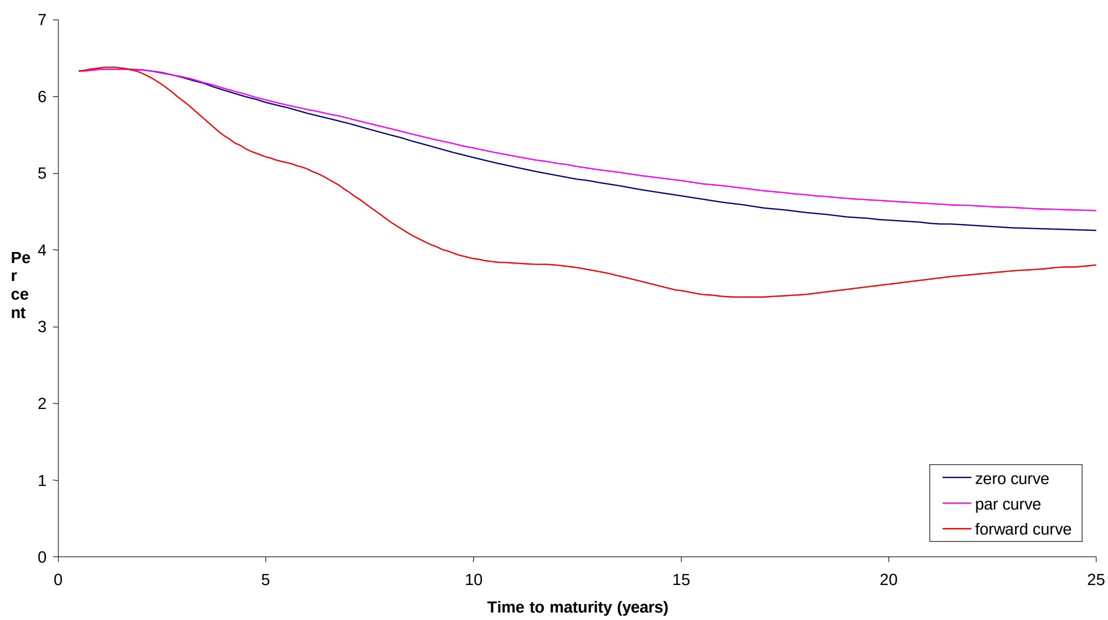



**Figure 2: Yield curves for 31 March 2000**




## Constructing the yield curve

If a market has liquid zero-coupon government bonds maturing at every future date, the yields on these could be used to construct the yield curve directly. With the existence of the UK strips market it is possible to observe the prices of over 50 traded zero-coupon bonds with maturities at six-month intervals right across the maturity spectrum. However, the strips market has grown slowly since its inception in December 1997 and suffers from low levels of liquidity. For example, by the end of June 2000 just £2.5 billion nominal (or 2.2%) of strippable gilts were held in stripped form and weekly turnover in gilt strips averaged around £50 million nominal, compared with around £30,000 million nominal in non-strips. This low level of liquidity means that the yields on these securities cannot be relied upon at present for the construction of the yield curve[1].

Instead, the yield curve must be estimated using the prices of coupon-paying bonds. This introduces problems of its own. For instance, conventional gilts are not equally spaced through the maturity spectrum - there are many “gaps” over which one needs to interpolate in order to construct a continuous yield curve. Also, the technical task of identifying the yield curve is further complicated by the existence of six-monthly interest payments.

**Choice of model**

In order to construct a continuous yield curve it is necessary to specify a functional form for the curve to be fitted to the data. When deciding which functional form to use it is important to consider the shapes that the yield curve should be allowed to take - in other words, what trade-off to make between the “smoothness” of the curve (removing “noise”, such as pricing anomalies, from the data) and its “responsiveness” (its flexibility to accommodate local changes in the shape of the curve). The purpose to which the yield curve is to be put is clearly relevant to this decision.

Whilst for macroeconomic analysis it may be desirable to fit a fairly simple function to the data in order to capture the general shape of the curve, for most of the analysis performed by the DMO it is important that the yield curve fits closely to the data, suggesting a more complex functional form.

The yield curve model employed by the DMO was originally developed by Mark Fisher, Douglas Nychka and David Zervos at the US Federal Reserve Board[2]. In common with many of the academic studies on yield curve modelling this model uses a cubic spline (or piecewise cubic function) for its functional form, giving the curve a high degree of flexibility. Intuitively, a cubic spline can be thought of as a number of separate cubic functions, joined “smoothly” at a number of so-called *join* or *knot* points. The greater the number of knot points the higher the degree of flexibility of the resultant curve.

In addition to specifying the number of knot points it is also necessary to decide on their location (ie. the maturities at which they should be located). Although the knots could be distributed evenly over time to maturity it is common to concentrate them towards the short end to capture the (typically) greater complexity of the curve at shorter maturities.

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[1] The same is true of Treasury bills.
[2] For more details on the “FNZ” model see paper 95-1 in the Federal Reserve Board’s Finance and Economics Discussion Series “Fitting the term structure of interest rates with smoothing splines”.

Although much of the early research on yield curves used regression splines, more recently several studies - including the FNZ paper - have used smoothing splines. Whilst for regression splines the number of parameters (or knot points) must be chosen exogenously, smoothing splines have a penalty function that penalises excess roughness (ie. oscillatory behaviour) in the curve and automatically determines the effective number of parameters. An increase in the penalty reduces the effective number of parameters. This means that the model allows the data to determine the appropriate number of parameters. In the DMO model, the extent to which the penalty function reduces oscillations in the fitted yield curve depends on the size of a parameter which is referred to as the *roughness penalty*. If this parameter was zero there would be no smoothing of the curve and the resulting forward curve could oscillate wildly. Alternatively, if it was large, the estimated forward curve would be inflexible and could be close to a straight line. The DMO determines the optimal value of the roughness penalty using a technique which is referred to as *generalised cross validation*, which is explained below.

Given a set of observations it is possible to fit numerous alternative curves through these points. The "goodness of fit" of each curve can be measured by taking an observation that was omitted from the estimation and measuring the difference between this observation and its estimated value implied by the curve[3]. The lower this difference the better the fitted curve. Since the choice of observation to omit is arbitrary, cross validation is employed to ensure a more rigorous approach. This technique avoids the problem of identifying which observation to exclude by looping over all the observations in turn, omitting each one and then fitting a curve. The differences between the omitted observations and the curve's estimated values are squared and added together to give an overall cross validation "score". Different values of the roughness penalty parameter give different scores, forming a function referred to as the cross validation function. The optimal value of the roughness penalty can then be found by minimising the cross validation function. However, fitting forward rate curves while repetitively omitting different observations makes standard cross validation a computationally expensive estimation procedure. Instead, the DMO employs a variant called generalised cross validation, which formulates the function to be minimised in a slightly simpler way in order to produce a more efficient solution.

Although generalised cross validation determines the effective number of parameters used, for a given run of the yield curve it is still necessary to specify an initial set of parameters from which to construct the optimal set. Fisher, Nychka and Zervos suggest choosing the number of knot points to be roughly one third of the sample size. With their sample size of between 160 and 180 bonds, applying this rule resulted in 50 to 60 knot points. With the much smaller number of bonds in the UK market, application of the "one third rule" means that the DMO's model currently uses 10 knot points. The maturities at which these knots are located are 0, 2, 4, 6, 8, 10, 12, 15, 20 and 40 years.

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[3] It is important that the observation used was not included in the estimation since otherwise this would lead to biased results.

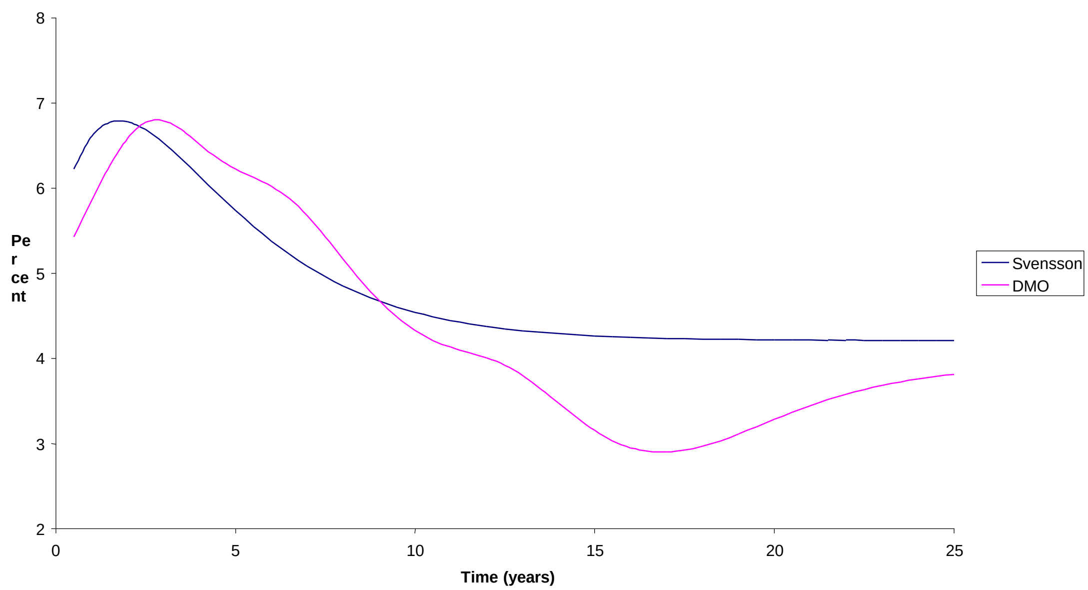
Another issue when constructing a spline based model is what form to use for the cubic spline equation itself. A cubic spline is usually defined to be a linear combination of underlying component or basis functions. Care is required when choosing the form of these component functions of the cubic spline since not all basis functions are equally capable of producing reliable estimates of the yield curve. When fitting the model to the data, some spline bases can result in inaccuracies arising from calculating the difference between large numbers. In keeping with the FNZ model, the DMO solves this problem by employing a basis of B-splines. These are functions which are identically zero over a large portion of the maturity spectrum and thus avoid the loss of accuracy introduced with other bases. Whilst for some yield curve models the function is fitted to the zero-coupon curve, the DMO's model fits to the implied forward rate curve.

Separate from the question of how flexible the model should be is the issue of whether the model should be constrained to produce asymptotically flat forward rates for long maturities. The argument for imposing such a constraint is the view that market participants are unlikely to have different expectations for the interest rate in 24 years' time from that in 25 years' time, for example. However, in practice observed yields do trend downwards at the long end in some markets. For instance, for several years now the supply/demand disequilibrium at the long end of the UK market has resulted in the longest dated gilts trading at a relative price premium (and hence lower yield) to other gilts.

Another less significant reason why the yield curve might - in practice - slope downward at the long end is because of the convex nature of long bonds. The *convexity* of a bond is a measure of the curvature of its price/yield relationship (ie. the degree to which the curve defining the relationship between a change in the bond's price and its corresponding change in yield diverges from a straight-line). In principle, a given bond will fall in price less than a less convex one when yields rise, and will rise in price more when yields fall, ie. convexity can be equated with the potential to outperform. Thus, other things being equal, the higher the convexity of a bond the more desirable it is to investors, and some investors may be prepared to accept a bond with a lower yield in order to gain convexity. Given that the longest dated gilts are considerably more convex than shorter dated securities this could lead to them trading at a premium to other gilts.

Again, the purpose to which the yield curve is to be put is relevant to whether a constraint should be imposed. Whilst it may be reasonable to impose an asymptotic constraint if the model is to be used to indicate the underlying interest rate expectations of market participants, for relative value analysis it is important that the curve accurately reflects the rates available in the market. With these factors in mind, the DMO's yield curve model is not constrained to flatten at the long end. Figure 3 compares the implied forward curve obtained from the DMO model for 25 October 1999 with that from the Svensson model[4]. The latter uses a simple functional form which is constrained to produce a flat forward curve at long maturities.

**Figure 3: Implied forward rate curves for 25 October 1999**




## Minimising yield or price errors

When fitting the yield curve model, the parameters of the model are estimated by minimising the errors between actual bond prices and the corresponding theoretical prices derived from the model. Minimising price errors sometimes results in fairly large errors for short maturity bonds since their prices are less sensitive to movements in yields than for longer maturity bonds[5]. The estimation of the short end of the curve can usually be improved by choosing the parameters to minimise yield errors instead, although this may lead to a slight deterioration in the fit of the curve at the long end. Rather than follow the original FNZ approach of minimising price errors, the DMO has modified the model to minimise price errors weighted with respect to the reciprocal of duration. Minimising duration-weighted price errors in this way is an approximation to minimising yield errors.

## Tax effects

Tax rules can materially affect the prices of bonds and, if their effects are ignored in the modelling process, can distort the estimate of the yield curve. Prior to April 1996, tax-paying investors in the gilt market were taxed on coupon income, but were exempt from taxation on capital gains. This led to a pronounced tax effect in the market as tax-paying investors bid up the prices of low-coupon bonds relative to high-coupon bonds. As a result of this, the

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[4] For more details see: Lars Svensson, "Estimating and interpreting forward interest rates: Sweden 1992-94", International Monetary Fund Working Paper No.114 (September 1994).
[5] For a given change in yield, the price of a short-dated bond will change much less than that of a longer-dated bond.

Bank of England - as the former government debt manager in the UK - employed a complex tax model when estimating the yield curve.

Under the tax regime effective from April 1996 this “coupon effect” largely disappeared, with just 3½% Funding 1999-2004 and 5½% Treasury Loan 2008-12 being grandfathered under the old regime. However, the new regime introduced a distortion of its own, albeit slight compared with that under the previous system. This arose because investors in strippable gilts were able to benefit from delayed tax payments on coupon income relative to those holding non-strippable bonds. In order to compensate for this when estimating the yield curve, the DMO employed a model which calculated, for each coupon of a given gilt, when the tax would be paid under quarterly accounting (the system for non-strippable gilts) and when it would be paid if the bond were strippable. The present value of the tax payments under both scenarios was then computed and the difference obtained.

As investors paying the difference between the payments (ie investors in non-strippable bonds) would have received some tax relief on the earlier payment which is spread over the life of the bond, the model made a further allowance for this. Since the tax treatment between strippable and non-strippable bonds was harmonised in April 1999 it is now no longer necessary to make a tax adjustment to the yield curve.

**Choice of which bonds to use in the estimation**

One of the important issues to consider when modelling the yield curve is which bonds to use in the estimation. To produce a meaningful measure of the government bond yield curve, only government bonds should be used since only they are normally regarded as being free from default risk. For example, the price of a corporate bond will typically be lower than that of a government bond of identical coupon and maturity, reflecting the credit risk of the corporate issuer.

In addition to conventional bonds, in many markets government bonds also exist with embedded optionality or with cash flows which are either linked to inflation (index-linked bonds) or that are periodically reset (floating rate securities). Bonds with embedded optionality give either the issuer or the holder some discretion to redeem early or to convert to another security. For example, several gilts are double-dated, giving the Treasury the option to redeem the bond at face value at any time between two dates specified at the time of issue. The embedded optionality will affect the valuation of such bonds relative to other bonds in the market. The extent to which the option will impact on a bond’s price depends on the market value of the option. Hence, in order to incorporate callable bonds successfully in the estimation of the yield curve it is necessary to build in an option pricing model. The additional complexity that this gives rise to means that in practice callable bonds are normally excluded from the yield curve estimation. This is the practice followed by the DMO.

Since the return on index-linked bonds is measured in real rather than nominal terms it is inappropriate to use them in the estimation of the nominal yield curve. There is currently only one floating rate bond in the UK market and since this only provides a measure of very short-term (ie 3 month) interest rates it too is excluded from the estimation.

Another selection criterion used when deciding which bonds to use in the yield curve is that of liquidity. For instance, a curve fitted to prices of bonds that are so illiquid that they rarely trade (and for which it may be difficult to obtain good quality prices) runs the risk of being mis-informative. As a result, illiquid bonds are often dropped from the estimation process. The simple proxy that the DMO uses to build an automatic liquidity criterion into the estimation procedure is to exclude all stocks of size below a given nominal amount outstanding. At present this nominal floor is set at £400 million – the same as the rump[6] threshold currently used by the DMO. The DMO also excludes bonds trading when-issued as well as all bonds with less than 3 months to maturity due to the difficulty of accurately estimating the curve at very short maturities[7]. A full list of the bonds currently used by the DMO to estimate the yield curve appears in the Appendix.

**Uses of the DMO yield curve model**

The DMO routinely runs its yield curve program at the end of each day, as well as occasionally running it on an intra-day basis. The DMO makes extensive use of the data from its model. For instance, the rates at which public corporations and local authorities can borrow from the Government[8] are determined from the par yield curve. These rates are usually published once a week, but following large market movements they are re-fixed on a more frequent basis.

The DMO also uses its model for internal monitoring of the value of individual bonds relative to the yield curve. The difference between the actual yield on a bond and its theoretical yield implied by the yield curve is referred to as the bond's cheap/dear residual or its *theoretical spread*. On a given day, the theoretical spread for a bond gives an indication of whether it is trading cheap (positive spread) or expensive (negative spread) relative to the yield curve.

Figure 4 illustrates the cheap/dear residuals for a range of gilts on a recent date. In addition to monitoring the absolute level of cheapness or dearness of individual bonds the DMO looks at how their cheapness/dearness has changed over time. Theoretical prices from the DMO's yield curve model will also have a role to play in any reverse auctions that the DMO undertakes. Reverse auctions will be of a multiple stock format and in order to rank the

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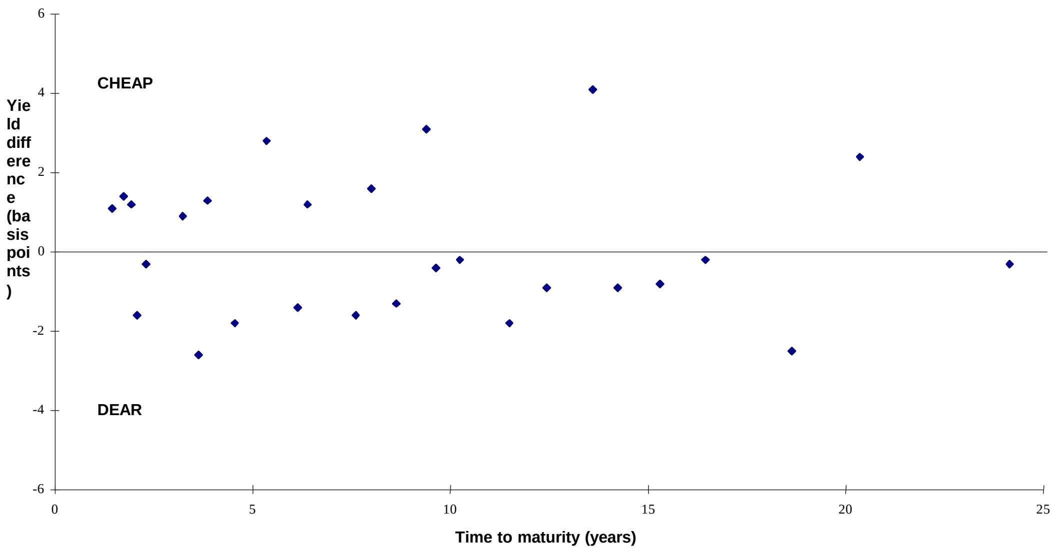
[6] Rump stocks are relatively small gilts (in terms of nominal outstanding), which GEMMs are not required to make a market in, but for which the DMO will be prepared to make a price if requested.

[7] This is partly due to the fact that a slight inaccuracy in the price can lead to a large yield error for short-dated bonds.

[8] National Loans Fund (NLF) and Public Works Loan Board (PWLB) rates.

bids the DMO will accept stock from the highest relative yields offered (as measured against the theoretical bond yields from the yield curve).

**Figure 4: Theoretical spreads of bonds against the yield curve**




The DMO also uses its yield curve model when setting the terms for gilt conversions. Conversion terms are decided by the DMO, using its yield curve model to provide a benchmark ratio for the offer. This benchmark ratio is calculated by valuing both the source and destination stocks by discounting each of the cash flows to the conversion date using the forward yield curve on the date of announcement of the conversion terms. The DMO then derives the published conversion ratio from this benchmark ratio by taking some account of the observed cheap/dear characteristics of the source and destination bonds.

## Appendix: Bonds used to estimate the yield curve as at 30 June 2000

8% TREASURY 2000
10% TREASURY 2001
7% TREASURY 2001
7% TREASURY 2002
9 ¾% TREASURY 2002
8% TREASURY 2003
10% TREASURY 2003
6 ½% TREASURY 2003
5% TREASURY 2004
6 ¾% TREASURY 2004
9 ½% CONVERSION 2005
8 ½% TREASURY 2005
7 ¾% TREASURY 2006
7 ½% TREASURY 2006
8 ½% TREASURY 2007
7 ¼% TREASURY 2007
9% TREASURY 2008
5 ¾% TREASURY 2009
6 ¼% TREASURY 2010
9% CONVERSION 2011
9% TREASURY 2012
8% TREASURY 2013
8% TREASURY 2015
8 ¾% TREASURY 2017
8% TREASURY 2021
6% TREASURY 2028
4 ¼% TREASURY 2032